Method for determining relaxation factor for modified Newton-Raphson method for non-linear systems

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Abstract — In this paper, we present an original algorithm to find the relaxation factor for a modified Newton-Raphson method in a faster way that uses fewer calculations. This method, based on the Functional NR principle suggested in [2], consists of minimising the energy functional. It decreases the number of computations linked to the relaxation coefficient, and ensures over-relaxation, in order to speed up the convergence of the solution. Finally, the performance of the suggested algorithm will be evaluated.

I. INTRODUCTION

The non-linear magnetic materials can lead to convergence difficulties in the context of simulation by finite elements of electromagnetic devices, leading to long computation times or even a total failure to converge. For these reasons, an efficient algorithm for determining relaxation factors involving few calculations is desirable, and serves as the motivation for the current research.

II. CONTEXT

In order to solve non-linear systems, the Newton-Raphson method is commonly used. Let $F(X)$ be a function defined on IR^n. The Newton method iteratively determines the zero of this function by using a Taylor series expansion of first degree. Around an arbitrary point X_k, this expansion is written as :

$$F\left(X_{k+1}\right) \cong F\left(X_k\right) + \left[\frac{\partial F}{\partial X^T}\right]_{X_k} \Delta X_k \qquad (1)$$

Newton's method builds a sequence of approximations by finding the zero of the Taylor expansion around the solution obtained from the previous iteration. Thus, the solution X_{k+1} of the iteration $k+1$ satisfies $F(X_{k+1}) = 0$, where $\Delta X_k = X_{k+1} - X_k$, $\partial F/\partial X^T$ is called the Jacobian matrix of the system and $F(X_k)$ vector is called the residual.

The approximation X_k gets even closer to the real solution as the residual decreases. Newton's method converges quickly when the Function $F(X)$ satisfies certain monotonic conditions and when the initial estimate is close enough to the real solution. But those conditions are rarely met. One solution is to relax the problem, i.e. set

$$X_{k+1} = X_k + \alpha \Delta X_k \quad \text{with} \quad 0 < \alpha \le 1 \qquad (2)$$

This relaxation factor is found with a line search. Two different approaches are compared: the Residual NR method [1] applying the dichotomy principle and Functional NR method minimising a functional to obtain an approximation of the optimal coefficient [2]. Finally, a third method will be discussed, derived from [3] with some improvements.

III. RESIDUAL NR - 1992

A. Principle

This method follows the dichotomy principle. First, the norm of the initial residual is computed, and then a second norm one using a relaxation factor $\alpha = 1$. While the residual norm decreases or the relaxation factor iteration is lower or equals to arbitrary, twelve, the relaxation factor is divided by two. When this norm increases, the chosen factor is the one from the previous iteration. If the norm increases over twelve successive iterations, the relaxation factor is set to 0.1. The cost of the residual computation, for a given α, being significant because of the assembling of the Finite Elements matrix, this method gets slower if the α minimising the residual norm is around zero.

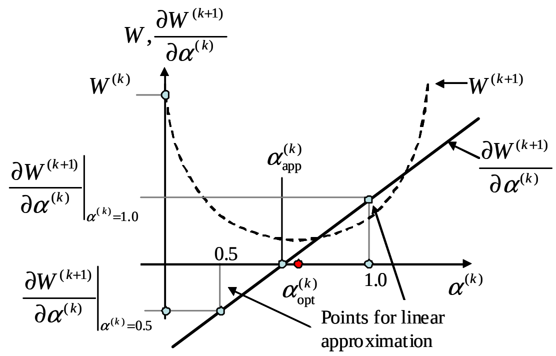

Fig. 1. *Functional* NR method for determining relaxation factor.

B. Analysis and Results

The method has been tested over a sample of diverse projects in 2D and 3D magnetostatics and magnetoharmonics which involve massive conductors, magnetic cuts, shell element regions [4] (resp. PB1, PB2 and PB3). It requires at least two, and a maximum of twelve, computations of the residual. The main interest is the robustness and speed of convergence. However, every residual computation is costly. As the relaxation factor computation represents 50% of the global runtime of a problem, a more effective search for optimal α could considerably reduce the total computation time. Over several projects, some « barrier zones » have been noticed (regions where the residual norm decreases very little between two NR iterations) which are caused by very small relaxation factors (≤ 0.1) which fail to perturb the solution enough.

IV. FUNCTIONAL NR - 2005

A. Principle

This method uses a whole new approach. Minimising the residual or the energy functional α is similar because, in the finite elements method, the system solution can be obtained by minimising this functional. Therefore, the equation to solve is $\partial\chi^{(k+1)}/\partial\alpha^{(k)} = 0$. Supposing that $\chi^{(k+1)}$ is quadratic, so $\partial\chi^{(k+1)}/\partial\alpha^{(k)} = 0$ is linear. An approximation of the derivative is given by :

$$\frac{\partial\chi^{(k+1)}}{\partial\alpha^{(k)}} = \sum_{i=1}^{\nu} G_i^{(k+1)}\delta x_i^{(k)} = 0 \quad (5)$$

where ν is the number of unknowns of the problem.

If $(\|G^{(k+1)}\|_2)^2$ has a parabolic shape on α_k, then $\partial(\|G^{(k+1)}\|_2)^2/\partial\alpha^{(k)}$ is a linear function and can be represented by using two arbitrary points of α_k. In the case where $(\|G^{(k+1)}\|_2)^2$ is not quadratic, it is possible to linearise $\partial(\|G^{(k+1)}\|_2)^2/\partial\alpha^{(k)}$. As shown on the figure (1), $W^{(k+1)}$ represents the objective function and corresponds to $(\|G^{(k+1)}\|_2)^2$. This method is called *Functional* NR. Moreover, it allows over-relaxation, i.e., take α greater than 1.0.

This method presents the advantage of computing residual for only two values of α, matching the minimum number of computations with Residual NR method. The disadvantage is the strong hypothesis over the parabolic shape of the squared residual norm. The following algorithm is suggested to overcome this difficulty.

B. Modified Functional NR

The Modified Functional NR (algorithm represented in Fig. 2) sets the relaxation factor to 1.0 at every first NR iteration. This stems from the fact that for 70% of the projects, for the first iteration, the squared residual norm is minimal for α = 1.0. This permits to save two residual computations, and then, time. For the other cases (30%), it also permits to get out « barrier zones » (solution increment really small).

Algorithm 1 Search for the relaxation factor with modified *Functional NR* for every NR iteration

```
if IT_NR = 1 then
  α_app = 1.0
else
  Computation of initial residual Y(α = 0) = RES^2
  Evaluation of Y and ∂(||G^(k+1)||_2)^2/∂α^(k) for α = 0.5 , 1.5
  Application hypothesis
  if Y(α = 0) > Y(α = 0.5) then
    Searching for the zero ⇒ α_app
  else
    α_app = 0.25
  endif
endif
```

Fig. 1. The Modified Functional NR algorithm.

The application hypothesis about the quadratic shape of the squared residual norm seems to validate 80% of the cases. The calculation of the derivative $\partial\chi/\partial\alpha$ gives an approximate optimal α.

When the hypothesis is not verified, the value of α is fixed at 0.25, in order to have a coefficient large enough (> 0.1) and consistent in relation to the shape of the curve of squared residuals.

C. Results

The table I lists, for three distinct electromagnetic problems, the number of NR iterations (**It NR**), number of residual computations (**C Res**) for every NR iteration, average time (**TM / IT R**) and total time (**TR**) for searching for the relaxation factor; for each previously presented methods.

TABLE I
COMPARATIVE PERFORMANCE

Test Case		Pb 1	Pb 2	Pb 3
It NR	Residual NR	12	8	22
	Modified Functional NR	7	7	8
C Res (sec)	Residual NR	24	16	117
	Modified Functional NR	12	12	14
TM / IT R (sec)	Residual NR	0.38	1.54	3.5
	Modified Functional NR	0.27	1.21	0.88
TR (sec)	Residual NR	4.6	12.3	77.0
	Modified Functional NR	1.9	8.5	7.1

For those three test cases, saving times concerning the coefficient search are consequent. This method allows to save Newton-Raphson iterations but mostly, time of relaxation factor search. The over-relaxation brings a certain convergence acceleration. 75% of the chosen relaxation coefficients are greater than 1.0.

V. CONCLUSION

We propose an efficient determining relaxation algorithm. In average over ten projects, the suggested method allows to save 30% of the overall resolution time, 6 Newton-Raphson iterations and 18 residual computations during the relaxation coefficient search. In order to strengthen the method robustness, it is possible to integrate other values of α to find a better approximation of alpha optimum. The Residual NR method could also be accelerated by decreasing the maximal iterations number and setting α = 0.25 when residual never decreases.

VI. REFERENCES

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